Computational Topology (Spring 2018) — Homework 5

- You **must email your submission** as a **PDF file** to bkrishna@math.wsu.edu. You are welcome to write answers by hand, and scan or take photos of the writings. Put all the images on a PDF file, though.
- Your file name should identify you. For instance, if you are Stan Jarsh, you should name your submission StanJarsh_Hw5.pdf. **Please start your name in this format. If you want to add more bits to the title, e.g., Math574, you could name it StanJarsh_Math574_Hw5.pdf, for instance. Also, please avoid white spaces in the file name :-).**
- **This homework is due before midnight on Thursday, March 29**.

1. (25) Let K triangulate an orientable 2-manifold (without boundary), and let $\mathrm{Sd}\,K$ denote its barycentric subdivision.
   (a) Show that the vertices of $\mathrm{Sd}\,K$ can be 3-colored, i.e., each vertex assigned one of three colors, such that no two neighboring vertices, i.e., vertices that share an edge, receive the same color.
   (b) Prove that the triangles in $\mathrm{Sd}\,K$ can be 2-colored such that no two triangles sharing an edge receive the same color.
2. (30) You are given a list of f faces (or triangles) using the n vertices $\{1, 2, \ldots, n\}$. The goal of this exercise is to propose efficient methods to check if the given simplicial complex triangulates a 2-manifold.
   (a) Describe an *efficient* method to test whether or not every edge is shared exactly by two triangles.
   (b) Describe an *efficient* method to test whether or not every vertex belongs to a set of triangles whose union is a disk.

   To be more precise, the book (by Edelsbrunner and Harer) asks you to describe procedures for these tasks that take $O(f + n)$ time, i.e., the total number of operations taken by each of the two procedures is proportional to $f + n$.
3. (15) Construct two topological spaces that have isomorphic homology groups, but are not homotopy equivalent.
4. (35) Find the Betti numbers $\beta_0, \beta_1, \beta_2$ of the Klein bottle over $\mathbb{Z}_2$. You could use the triangulation of the Klein bottle introduced in Lecture 6. Which other 2-manifolds introduced in class have the same set of Betti numbers over $\mathbb{Z}_2$? (You are welcome to use the code in the following question for this problem as well, where applicable.)
5. (60) Write a function in Octave or another similar package/language that takes as input a matrix with entries in $\{0, 1\}$, and finds its Smith normal form (SNF) when working over $\mathbb{Z}_2$.

   Use the above function to compute all relevant Betti numbers of the *patched* triangulation of the spine seen in Homework 3, i.e., after you closed the holes by adding three triangles.