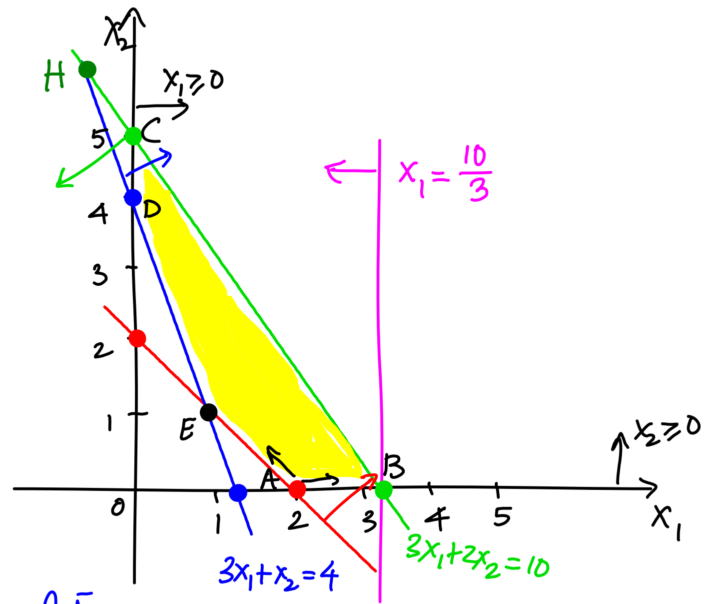
MATH464 – Lecture 11 (02/13/2018)

Today:
* more on degeneracy
* properties of polyhedra
* ~~Simplex method~~

Def A basic solution $\bar{x}$ of P in standard form is **degenerate** if more than $n-m$ x_j's are zero.



$$
\begin{aligned}
x_1 + x_2 - x_3 \quad &= 2\\
3x_1 + x_2 \quad - x_4 \quad &= 4\\
3x_1 + 2x_2 \qquad + x_5 &= 10\\
x_1 \qquad\qquad + x_6 &= 10/3
\end{aligned}
$$

$x_j \geqslant 0 \;\; \forall j$

$$A = \left[\begin{array}{cccccc} 1 & 1 & -1 & 0 & 0 & 0\\ 3 & 1 & 0 & -1 & 0 & 0\\ 3 & 2 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]_{4\times 6} \quad \bar{b} = \begin{bmatrix} 2\\4\\10\\10/3 \end{bmatrix} \qquad \mathrm{rank}(A) = m = 4.$$

original A (the first three rows, first five columns); original $\bar{b}$ (the first three entries)

Indeed, $\mathrm{rank}(A) = 4 = m$ now. So we are still in the original setting of standard form polyhedron. Let's figure out the bfs corresponding to the vertex B. Notice that $\{x_1, x_3, x_4\}$ are all basic, i.e., >0 at B. We need one more x_j out of x_2, x_5, x_6) to complete a basis. Which one could we select?

With $B(1)=1$, $B(2)=2$, $B(3)=3$, $B(4)=4$, we get $B = \begin{bmatrix} 1 & 1 & -1 & 0\\ 3 & 1 & 0 & -1\\ 3 & 2 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix}$, $\det(B) = -2$,

So B is indeed invertible. We get $\bar{x}_B = \begin{bmatrix} x_1\\x_2\\x_3\\x_4 \end{bmatrix} = B^{-1}\bar{b} = \begin{bmatrix} 10/3\\0\\4/3\\6 \end{bmatrix}$.

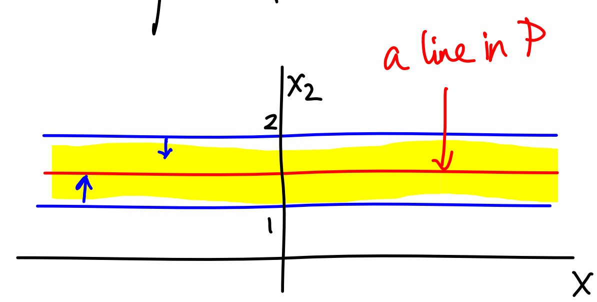
In fact, we could pick any one of $\{x_2, x_5, x_6\}$ with $\{x_1, x_3, x_4\}$ to get a basis here. Check the Octave session for details. In each case, we get 3 x_j's set at 0. Recall that $n-m = 6-4 = 2$ here, thus showing
including the non-basic x_j's
that more than $n-m$ x_j's are zero at the degenerate bfs.

Qn. If there are $k > n$ active constraints, is it always true that there are $\binom{k}{n-k} = \binom{k}{n}$ different bases that lead to the same degenerate bfs? (will be in next homework)

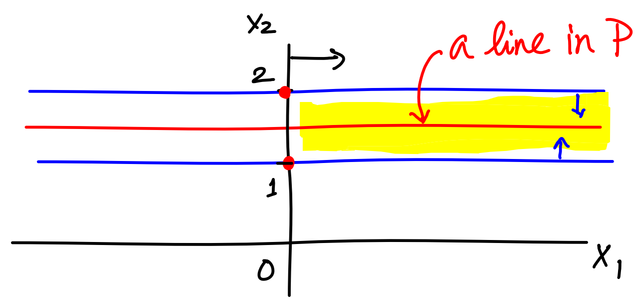
We have seen that vertices were optimal in many of the LP instances we solved. Indeed, we are building up the machinery to talk about the Simplex algorithm, which will move from one bfs to an adjacent bfs in each step. But we first study some properties of the polyhedron in general.

Does every polyhedron have an extreme point?

$P = \{\bar{x} \in \mathbb{R}^2 \mid 1 \leq x_2 \leq 2\}$ has no corner points.

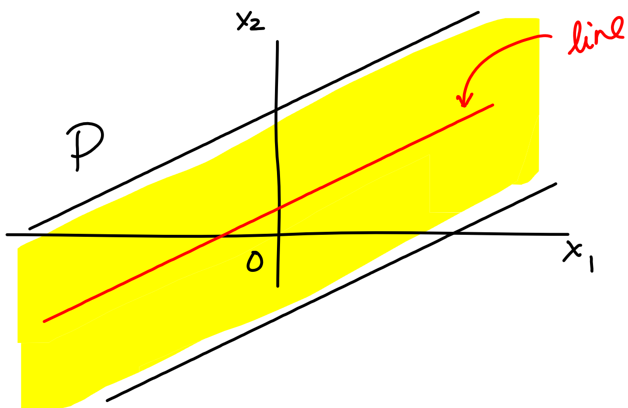


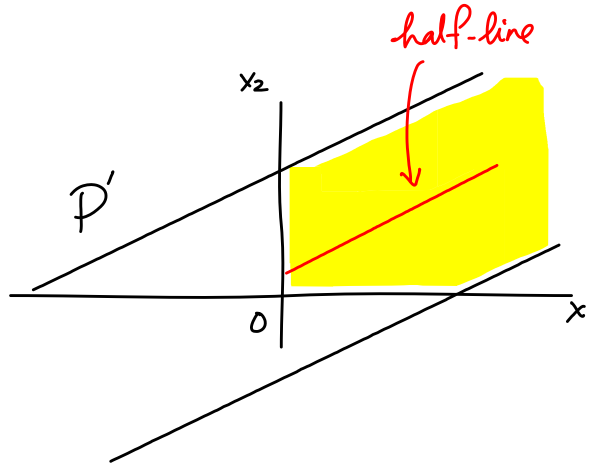
But if we were to add $x_1 \geq 0$, we get two corner points.



$P = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \geq \bar{b}\}$ with $A_{m \times n}$ and $m < n$ cannot have any basic solutions, and hence cannot have any bfs's!

Intuitively, P is "corner-free" if it contains a line, which extends without limit in either direction. For instance, P on the left has a line, and hence has no vertices.

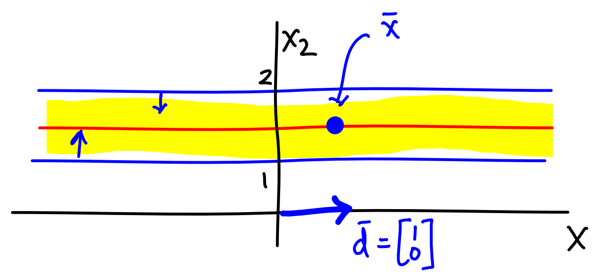




But P does not contain a line once $x_j \geq 0$ is added (to get P').

We formalize the notion of a set containing a line, and the intuition that a polyhedron without a line has a vertex.

Def $P \subset \mathbb{R}^n$ contains a line if there exists some $\bar{x} \in P$ and a direction $\bar{d} \in \mathbb{R}^n$, $\bar{d} \neq \bar{0}$, such that $\bar{x} + \lambda \bar{d} \in P \;\forall\, \lambda \in \mathbb{R}$.



(Theorem 2.6 BT-ILO) Let $P = \{\bar{x} \in \mathbb{R}^n \mid \bar{a}_i^T \bar{x} \geq b_i,\ i = 1, \ldots, m\}$, $P \neq \emptyset$. $m \geq n$

The following statements are equivalent.

(i) P has at least one extreme point.

(ii) P does not contain a line.

(iii) There are n vectors in $\{\bar{a}_1, \ldots, \bar{a}_m\}$ which are LI.

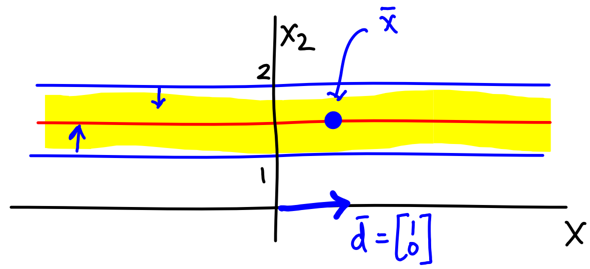
We had already noted that if $m < n$, P has no bfs.

We immediately get the following corollary.

BT-ILO Corollary 2.2 Every nonempty bounded polyhedron, and every polyhedron in standard form, has at least one bfs.

If P is bounded, it cannot contain a line, so (ii) holds. Notice that $\bar{a}_i^T \bar{x} \geq b_i$, $i=1,\ldots,m$ above include $x_j \geq 0 \; \forall j$. Indeed, as soon as we have added $x_j \geq 0 \; \forall j$, we do have n $\bar{a}_i$'s that are LI — the unit vectors corresponding to $x_j \geq 0$, i.e., $\bar{a}_i = \bar{e}_i$, the i^{th} unit vector. Hence (iii) holds.

Qn. If P has no corner points, does the LP
$$\min \{\bar{c}^T \bar{x} \mid \bar{x} \in P\}$$
have any optimal solutions?



e.g., $\min \{x_2 \mid 1 \leq x_2 \leq 2\}$ has alternative optimal solutions (any point on the line $x_2 = 1$ is optimal).

But $\min \{x_1 + x_2 \mid 1 \leq x_2 \leq 2\}$ is an unbounded LP.

At the same time, we could make the following statement. If P has no corner points, then the LP **cannot** have a unique optimal solution. The next theorem formalizes the reverse implication.

BT-ILO Theorem 2.7 Consider $\min\{\bar{c}^T\bar{x} \mid \bar{x}\in P\}$.
If P has at least one extreme point, and if the LP has an optimal solution, then there exists an extreme point which is an optimal solution.

We could get a slightly more general result, which specifies what happens when there is no optimal solution.

BT-ILO Theorem 2.8 Consider $\min\{\bar{c}^T\bar{x} \mid \bar{x}\in P\}$, where P is a polyhedron with at least one extreme point. Then the optimal cost is $-\infty$ or there exists an extreme point which is optimal.

See BT-ILO for the proofs.

Notice that the above theorems cover the case of polyhedra with at least one extreme point. But what about polyhedra without extreme points? Indeed, we can generalize the above theorem to get the following corollary. In particular, recall $\min\{x_2 \mid 1\le x_2\le 2\}$.

(Corollary 2.3 BT-ILO) Consider $\min\{\bar{c}^T\bar{x} \mid \bar{x}\in P\}$, where P is a nonempty polyhedron. Then the optimal cost is $-\infty$, or the exists an optimal solution.

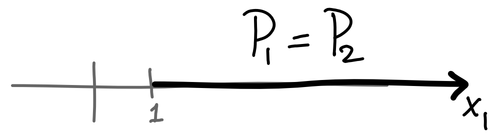
This result might not hold for non-linear problems.

consider $\left\{\begin{array}{l}\min \frac{1}{x}\\ \text{s.t. } x\ge 1\end{array}\right\}$ There is no optimal solution here, but the optimal cost is not $-\infty$.

We are now ready to present the simplex method to solve LPs in general dimensions. We will present this algorithm for LPs in standard form. In this context, we consider one more aspect of the standard form.

Qn. Does the "shape" of a polyhedron change when we convert it to standard form?

Consider $P_1 = \{x_1 \in \mathbb{R} \mid x_1 \geq 1\}$.

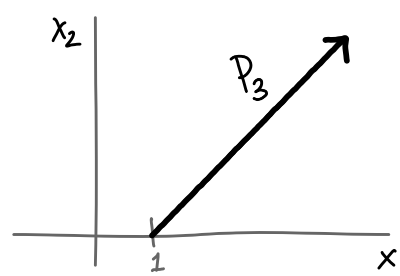


We add nonnegativity to get

$P_2 = \{x_1 \in \mathbb{R} \mid x_1 \geq 1, x_1 \geq 0\}$. Notice that $P_2 = P_1$.

We convert P_2 to standard form to get

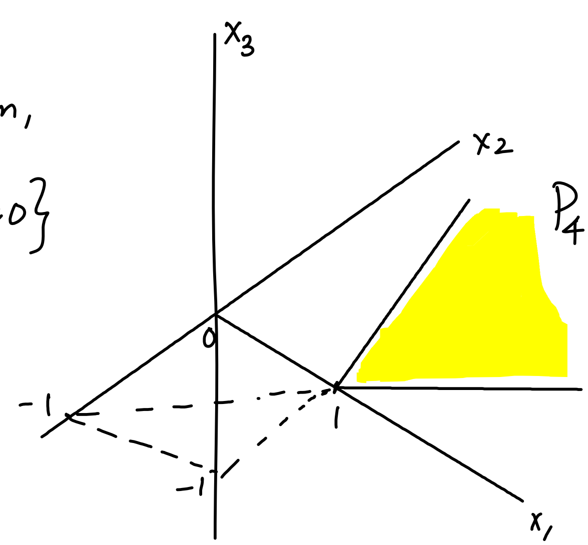
$P_3 = \{\bar{x} \in \mathbb{R}^2 \mid x_1 - x_2 = 1, x_1, x_2 \geq 0\}$



P_3 looks very much like P_1 (or P_2). They are all half-lines (or rays).

Now, if we convert P_1 to standard form,

we get $P_4 = \{\bar{x} \in \mathbb{R}^3 \mid \underbrace{(x_1 - x_2)}_{\substack{= x_1 \text{ in } P_1 \\ x_1 \text{ urs}}} - x_3 = 1, x_j \geq 0\}$



P_4 is a portion of a plane in the nonnegative orthant in $\mathbb{R}^3$. Notice the similarity to P_1, P_2, and P_3.