MATH464 – Lecture 13 (02/20/2018)

Today: * Problems from Hw6
* LP optimality conditions

Hints for Problems from Hw6

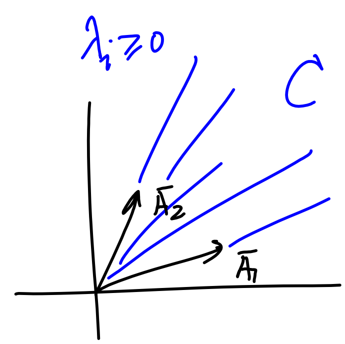
BT-ILO 2.6 Remember bfs for $P = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{b}, \bar{x} \geq \bar{0}\}$:

$A \in \mathbb{R}^{m \times n}$, rank $(A) = m$, $m \leq n$.

$\bar{x}$ is a bfs $P \Rightarrow$ $n-m$ of the x_j's are zero.

Also, recall that P (in standard form) must have at least one bfs.

C is the cone of $\bar{A}_1, \dots, \bar{A}_n$:



For any $\bar{y} \in C$, consider

$$\Lambda = \{\bar{\lambda} \in \mathbb{R}^n \mid A\bar{\lambda} = \bar{y}, \bar{\lambda} \geq \bar{0}\}$$

$\equiv \bar{b}$

Λ is a polyhedron in standard form $\Rightarrow$ has a bfs, which has $n-m$ entries $= 0$ (details to be provided).

(b) Use similar ideas about existence of a bfs for a polyhedron in standard form (with one extra constraint $\sum_{i=1}^{n_1} \lambda_i = 1$).

2.9 (a) Two different bases $\mathcal{B}$, $\mathcal{B}'$

$\mathcal{B} = \{B(1), \ldots, B(m)\}$, $\mathcal{B}' = \{B'(1), \ldots, B'(m)\}$

indices of the basis

$B(i), B'(i) \in \{1, \ldots, n\}$

B is the basis matrix

$n-m$ nonbasic variables are zero in a basic solution.

A basic solution is degenerate if more than $n-m$ x_j's are zero.

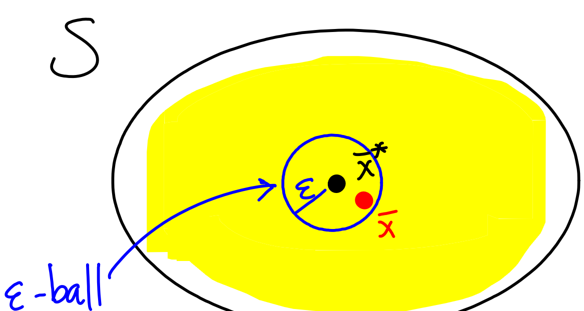
2.11 and 2.9 (c) are related.

2.16. P in standard form is $\{\bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{b}, \bar{x} \geq \bar{0}\}$.

The given set (as described) is obviously not in standard form. The question is whether this set could describe a set that is in standard form.

Hint: think P in standard form has a bfs, cannot contain a line, etc.

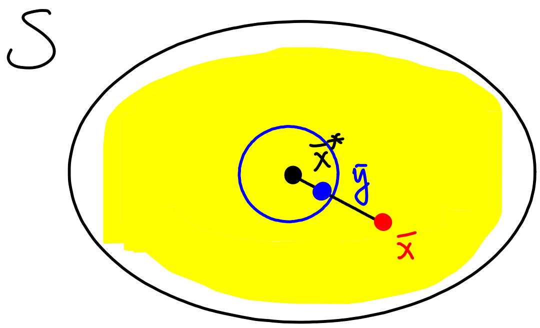
3.1.



$\bar{x}^*$ is a local optimum of $f \Rightarrow$ $\exists\ \varepsilon > 0$, s.t.

$f(\bar{x}^*) \leq f(\bar{x})\ \forall \bar{x}$ s.t. $\|\bar{x} - \bar{x}^*\| \leq \varepsilon$

Need to prove $\bar{x}^*$ is global minimum, i.e., $f(\bar{x}^*) \leq f(\bar{x})\ \forall \bar{x} \in S$ (and not just inside the ε-ball around $\bar{x}^*$).



Proof by contradiction:

Assume $\exists\ \bar{x}$ outside the ε-ball centered at $\bar{x}^*$ with $f(\bar{x}) < f(\bar{x}^*)$; i.e., $\bar{x}^*$ is not a global minimum.

S is convex $\Rightarrow$ $\bar{y} \in S$.

f is convex $\Rightarrow$ $f(\overbrace{\lambda \bar{x}^* + (1-\lambda)\bar{x}}^{\bar{y}}) \leq \lambda f(\bar{x}^*) + (1-\lambda) f(\bar{x})$

Use these inequalities to arrive at a contradiction.

The main point to remember is that you should be able to answer all questions using the results we described in class!

LP optimality Conditions

Recall: reduced costs: $\bar{c}'^T = \bar{c}^T - \bar{c}_B^T B^{-1} A$
for an LP in standard form: $\min\{\bar{c}^T\bar{x} \mid A\bar{x} = \bar{b}, \bar{x} \geqslant \bar{0}\}$ with
$A \in \mathbb{R}^{m\times n}$, $m \leq n$, $\text{rank}(A) = m$.

BT-ILO Theorem 3.1 $\bar{x}$ is a bfs, B: basis matrix, $\bar{c}'$ reduced costs.

(a) If $\bar{c}' \geqslant \bar{0}$, then $\bar{x}$ is optimal

(b) If $\bar{x}$ is nondegenerate and optimal, then $\bar{c}' \geqslant \bar{0}$.

Proof (a) Let $\bar{c}'^T = \bar{c}^T - \bar{c}_B^T B^{-1} A \geqslant \bar{0}^T$. We want to show
$\bar{c}^T\bar{x} \leq \bar{c}^T\bar{y} \;\; \forall \bar{y} \in P$ (i.e., $\bar{x}$ is optimal).

Let $\bar{y} \in P$ be an arbitrary feasible point. So, $A\bar{y} = \bar{b}, \bar{y} \geqslant \bar{0}$.

Let $\bar{d} = \bar{y} - \bar{x}$. Also, $A\bar{x} = \bar{b}$, $\bar{x} \geqslant \bar{0}$ (as $\bar{x}$ is a bfs).

Want to prove $\bar{c}^T\bar{d} \geqslant \bar{0}$

$\Rightarrow A\bar{d} = A(\bar{y} - \bar{x}) = \bar{0}$.

$\Rightarrow B\bar{d}_B + N\bar{d}_N = \bar{0} \quad \Rightarrow B^{-1}\left(B\bar{d}_B + \sum_{i \in \mathcal{N}} A_i d_i = \bar{0}\right)$.

$\mathcal{N}$ is the set of non-basic indices ($x_j = 0 \;\; \forall j \in \mathcal{N}$).

$\mathcal{B}$ is the set of basis indices, i.e., $\mathcal{B} = \{B(1), \ldots, B(m)\}$.

So, $\mathcal{B} \cup \mathcal{N} = \{1, \ldots, n\}$.

$$\Rightarrow \bar{d}_B + \sum_{i \in \mathcal{N}} B^{-1}A_i d_i = \bar{0} \quad \Rightarrow \bar{d}_B = -\sum_{i \in \mathcal{N}} B^{-1}A_i d_i.$$

So, $\bar{c}^T\bar{d} = \bar{c}_B^T\bar{d}_B + \bar{c}_N^T\bar{d}_N = \bar{c}_B^T\bar{d}_B + \sum_{i\in N} c_i d_i$

$$= \sum_{i\in N} c_i d_i - \sum_{i\in N} \bar{c}_B^T B^{-1} A_i d_i \qquad \bar{d}_B = -\sum_{i\in N} B^{-1}A_i d_i$$

$$= \sum_{i\in N} (c_i - \bar{c}_B^T B^{-1} A_i) d_i$$

$$= \sum_{i\in N} \underbrace{c_i'}_{\geq 0} d_i, \quad \text{as } c_i' = c_i - \bar{c}_B^T B^{-1} A_i, \text{ the } i^{th} \text{ reduced cost.}$$

We will be done if we can show $d_i \geq 0 \ \forall\, i \in N$, as $c_i' \geq 0$ is already given, and then we get $\bar{c}^T\bar{d} \geq 0 \Rightarrow \bar{c}^T\bar{y} \geq \bar{c}^T\bar{x}$.

We have $d_i = y_i - x_i \ \forall i \in N$. But $x_i = 0 \ \forall i \in N$ (as $\bar{x}$ is a bfs).

Also, $y_i \geq 0$, as $\bar{y} \in P$ and hence $\bar{y} \geq \bar{0}$.

$\Rightarrow d_i \geq 0 \ \forall\, i \in N. \Rightarrow \bar{c}^T\bar{d} \geq 0$, i.e., $\bar{c}^T\bar{x} \leq \bar{c}^T\bar{y}$.

$\Rightarrow \bar{x}$ is optimal.

Check BT-ILO for proof of statement (b).

$\square$

Equivalent definition of optimality conditions

→ we combine feasibility and optimality

A basis matrix B is optimal if

(a) $B^{-1}\bar{b} \geq \bar{0}$, and (feasibity)

(b) $\bar{c}'^{T} = \bar{c}^{T} - \bar{c}_B^{T} B^{-1} A \geq \bar{0}$ (optimality).

At $E(1,1)$, basis is

$\mathcal{B} = \{B(1), B(2), B(3)\} = \{1, 2, 5\}$

$B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ $B^{-1}\bar{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \geq \bar{0}$

$\bar{c}' = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} \geq \bar{0}$. So, E is optimal.