Convex Models in Bundle Methods for Nonsmooth Nonconvex Minimization: Prerequisite for a VU-algorithm

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Introduction

$\min_{x \in \mathbb{R}^n} f(x)$; f locally Lipschitz,
only one (Clarke) gradient $g(x)$,
computed by a black box at each x.

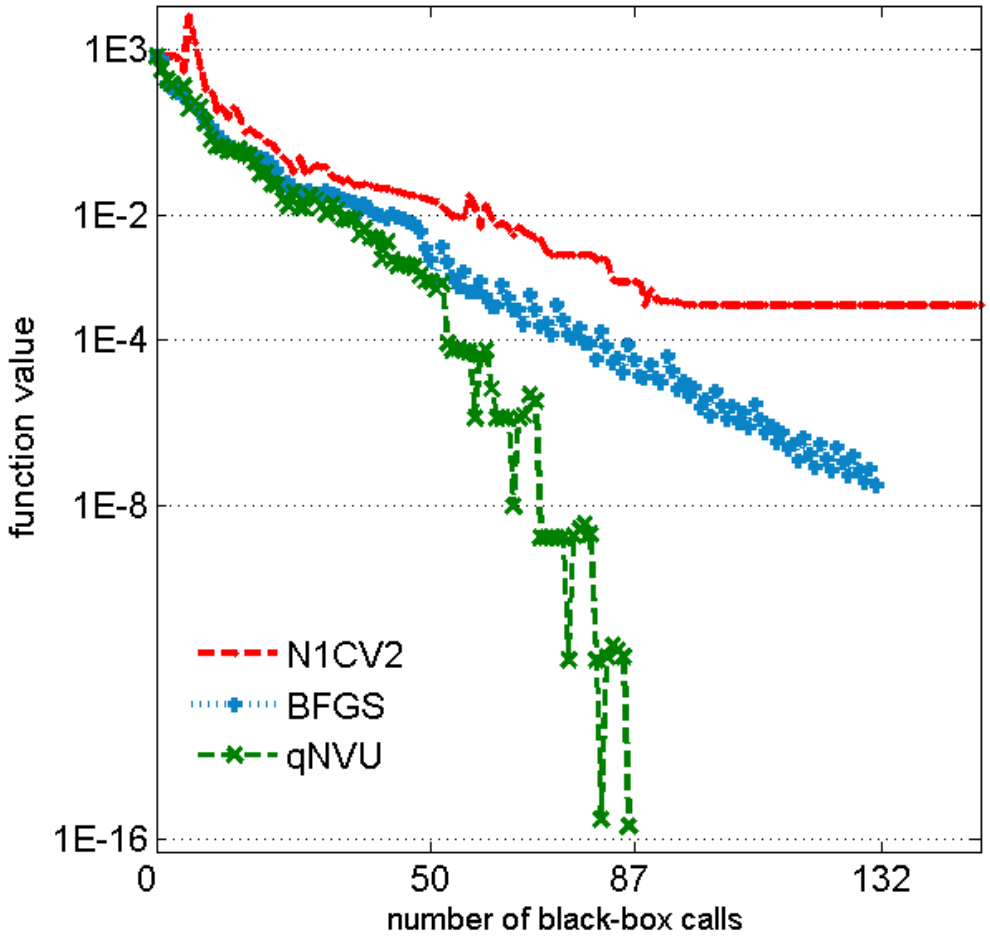
Ultimate Goal: Design a VU-algorithm for such f.

A VU-algorithm implicitly exploits underlying nonsmooth/smooth structure to achieve rapid convergence.

CONVEX CASE

Lewis and Overton 8-variable half-and-half function

[MS, A VU-algorithm for convex minimization, Math. Prog. 104(2-3), 583-608, 2005]



Sublinear, linear, and superlinear convergence - convex case

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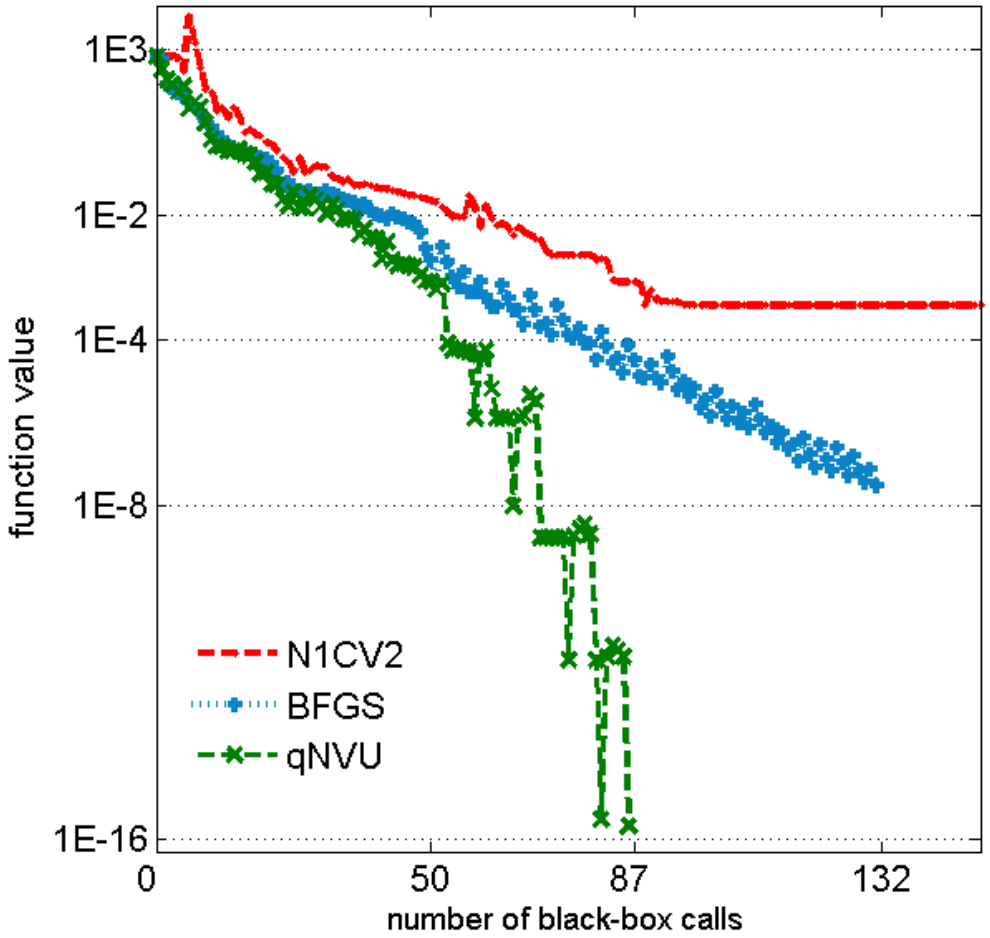

Sublinear, linear, and superlinear convergence -

What can be done for nonconvex case?

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Ultimate Goal: Design a VU-algorithm for such f.

A VU-algorithm implicitly exploits underlying nonsmooth/smooth structure to achieve rapid convergence. To do so, a suitable V-model for f is needed

This talk: Define a bundle method that achieves convergence to stationary points and produces good V-models for f. We refer to this as a viable algorithm with a viable V-model (i.e. polyhedral model). Also consider more general convex models.

Outline

- Difficulty on V-space due to nonconvexity
- Four general conditions for a viable bundle algorithm
- Framework with model functions M, centers x, line searches generating null or serious (next center) points
- General stationarity theorem
- Specific V-model bundle algorithm with safeguarded negative curvature corrections generated by line searches
- Specific null and serious point definitions for obtaining finite line searches for semismooth objectives and viability to imply asymptotic stationarity

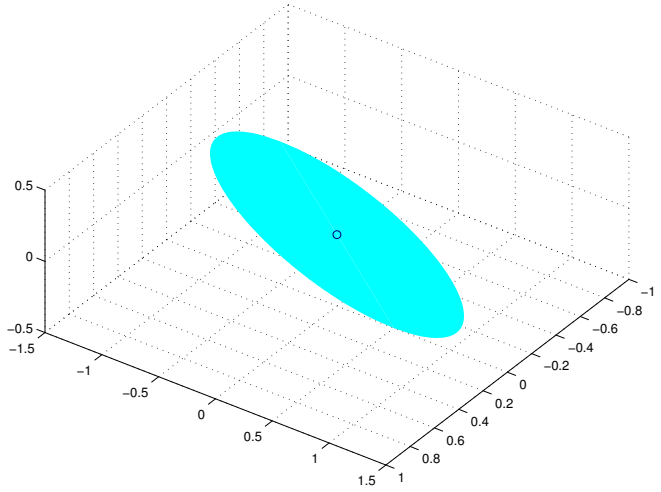
- **Future work for a complete VU-algorithm**

$\mathcal{V}$ and $\mathcal{U}$ subspaces and graph of f on $\mathcal{V}$

A nonconvex pdg-structured example

$$f(x_1, x_2, x_3) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}\ln\left(1 + \sqrt{(x_1^2 - 2x_2)^2 + (x_3 - x_2)^2}\right)$$

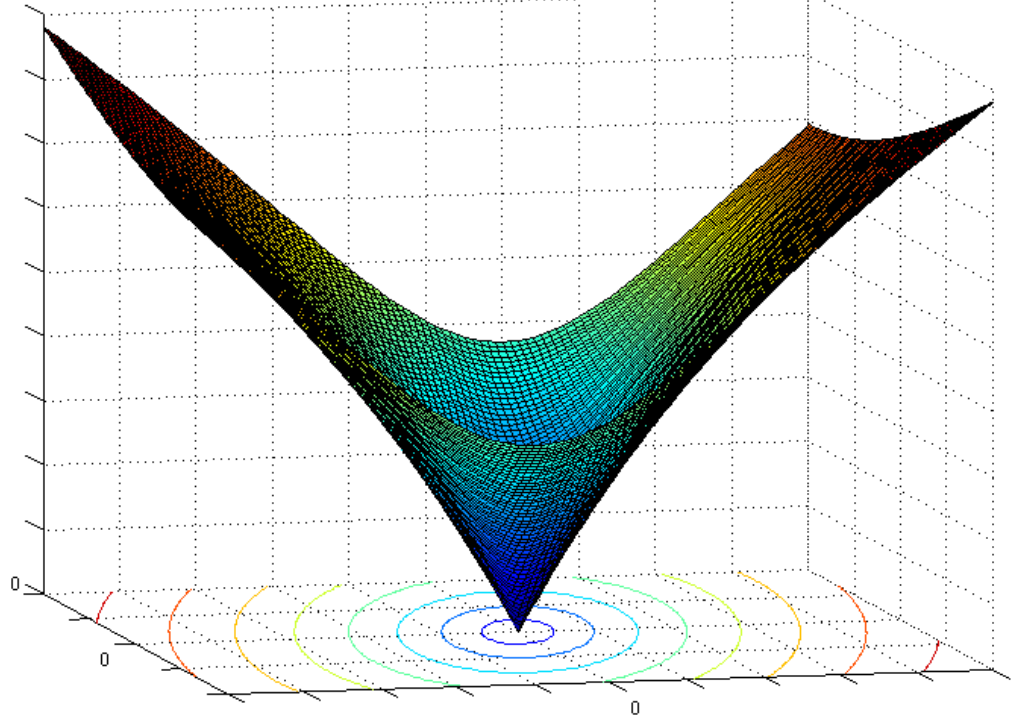
$x^* = (0, 0, 0)$ is a stationary point (minimizer)
zero subgradient $\in \partial f(x^*)$



In general, for any $\bar{x}$

$\bar{g} \in \partial f(\bar{x})$, $\quad \mathcal{V}(\bar{x}) := lin(\partial f(\bar{x}) - \bar{g})$ **and** $\mathcal{U}(\bar{x}) := \mathcal{V}(\bar{x})^{\perp}$

A view of f on $\mathcal{V}(x^*)$



Bundle iteration elements

For a prox-parameter $\mu > 0$ and a convex
model function $M (\approx f$ near a center $x)$ define

search direction $d(x) := \arg\min M(x + \cdot) + \frac{1}{2}\mu|\cdot|^2$,

aggregate gradient $G(x) := -\mu d(x) \in \partial M(x + d(x))$,

aggregate error $E(x) := M(x) - M(x + d(x)) - \mu|d(x)|^2 \geq 0$
(nonnegative by subgradient inequality for convex M),

progress measure $D(x) := f(x) - M(x + d(x)) - \frac{\mu}{2}|d(x)|^2$,

$$D(x) = f(x) - M(x) + E(x) + \frac{1}{2\mu}|G(x)|^2,$$

so $$D(x) \geq E(x) + \frac{1}{2\mu}|G(x)|^2 \geq 0$$

if $f(x) \geq M(x)$

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What is a viable model?

Viable Models

A model function M is viable if it is convex and satisfies

V1 The model is lower at its center x: $M(x) \leq f(x)$

V2 **Finite version**
Zero aggregate error implies
aggregate gradient is an f-subgradient:
$E(x) = 0 \implies G(x) \in \partial f(x)$

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For f convex, a cutting-plane model ensures both conditions hold, even an asymptotic version of **V2**.

Viable Algorithms

A bundle algorithm is viable if its model M is viable and it satisfies asymptotic conditions **V2** and **V3** and line search condition **V4**, depending on D-decreasing null and f-decreasing serious point definitions:

V2 **Algorithmic (asymptotic) version**

Zero asymptotic aggregate error implies associated asymptotic aggregate gradient is an f-subgradient:

$x_k \to \bar{x}$ and $E(x_k) \to 0 \Longrightarrow G(x_k) \to \in \partial f(\bar{x})$

V1+V2 and associated $\mathbf{D(x_k) \to 0}$ imply stationarity of $\bar{x}$

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V3 Zero asymptotic progress measure:
$x_k \to \bar{x} \Longrightarrow D(x_k) \to 0$

V4 **viable** line search at each iteration:
defined next to find a null or serious point

Bundle algorithm with viable line search

Input null/serious point defs. and $x_0, \mu_0 > 0, M_0$;
initialize $\ell := 0, k := 0, x := x_0$

Loop: Solve subproblem with x, μ_ℓ, M_ℓ for $d_\ell(x)$, $D_\ell(x)$.

If $D_\ell(x) = 0$, stop with x stationary.

Else, call for a line search from x along $d_\ell(x)$
with stepsize $t > 0$ so that ...

Bundle Algorithm with viable line search

either $t \uparrow \infty$ and $f(x + td_\ell(x)) \downarrow -\infty$
or it stops with $t = t_\ell$ such that the point

$$y_{\ell+1} := x + t_\ell d_\ell(x)$$

is either null or serious.

If $y_{\ell+1}$ is serious, set $x_{k+1} := y_{\ell+1}$, $\ell(k) := \ell$ and replace x by x_{k+1} and k by $k+1$.

Choose $\mu_{\ell+1} > 0, M_{\ell+1}$ based on bundled M_ℓ-data, $y_{\ell+1}$ and other data generated at iteration ℓ.
Replace ℓ by $\ell + 1$ and go to Loop.

Theorem(Stationarity).

Suppose

- the bundle algorithm does not terminate,
- the prox-parameters are in a positive interval $[\mu_{min}, \mu_{max}]$, and
- the progress measure sequence $\{D_\ell(x_k)\}$ is bounded.

If conditions **V1** to **V4** hold then any $\bar{x}$ that is a limit point of $\{x_k\}$ is stationary for f.
If x_k is finite with $\bar{x}$ being the last x_k then **V3** is written $D_\ell(\bar{x}) \to 0$ instead of $D_{\ell(k)}(x_k) \to 0$.

Now, for nonconvex f, we define a specific algorithm with viable polyhedral model, line search and null/serious definitions.

Specific viable model for nonconvex f

Polyhedral (V-model) function

$$M(x+d) = \max\{f(x) - \tilde{e}(x, y_i) + \langle \tilde{g}(x, y_i), d \rangle : y_i \in B\}\,.$$

Gradient estimates $\tilde{g}(x, y_i)$ and linearization error estimates $\tilde{e}(x, y_i)$ depend on the center x, a bundle B of previous iterates y_i and associated data

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V1 ensured by forcing $\tilde{e}(x, y_i) \geq 0$ via sufficient curvature terms or safeguards

Also beneficial to keep center x in B and to have $\tilde{e}(x, x) = 0$, $\tilde{g}(x, x) = g(x)$

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$$(y_i, f(y_i), g(y_i), \mathbf{H(y_i)}, \mathbf{s(y_i)})$$

with low rank **Hessian** matrix and **safeguard** scalar for down-shifting (both zero if f is convex)

V1 ensured by forcing $\tilde{e}(x, y_i) \geq 0$ via sufficient curvature terms or safeguards

Also beneficial to keep center x in B and to have the center linearization $f(x) + \langle g(x), d\rangle$ in the model-max

Convex f

$\tilde{g}(x,y) := g(y)$, independent of $x \neq y$,
$\tilde{e}(x,y) := e(x,y) := f(x) - (f(y) + \langle g(y), (x-y) \rangle)$;
with $y = y_i$ this gives lower cutting planes: $e(x, y_i) \geq 0$
and further convex analysis gives **V2**

Nonconvex f

Keep M polyhedral and simply modify $\tilde{e}, \tilde{g}$ via negative curvature corrections from $\mathbf{H}$, computed during line search. When corrections are not large enough there is a safeguard using $\mathbf{s}$ to make $\tilde{e}$ large enough to obtain **V2**, depending on outer semicontinuity of $\partial f(\cdot)$.

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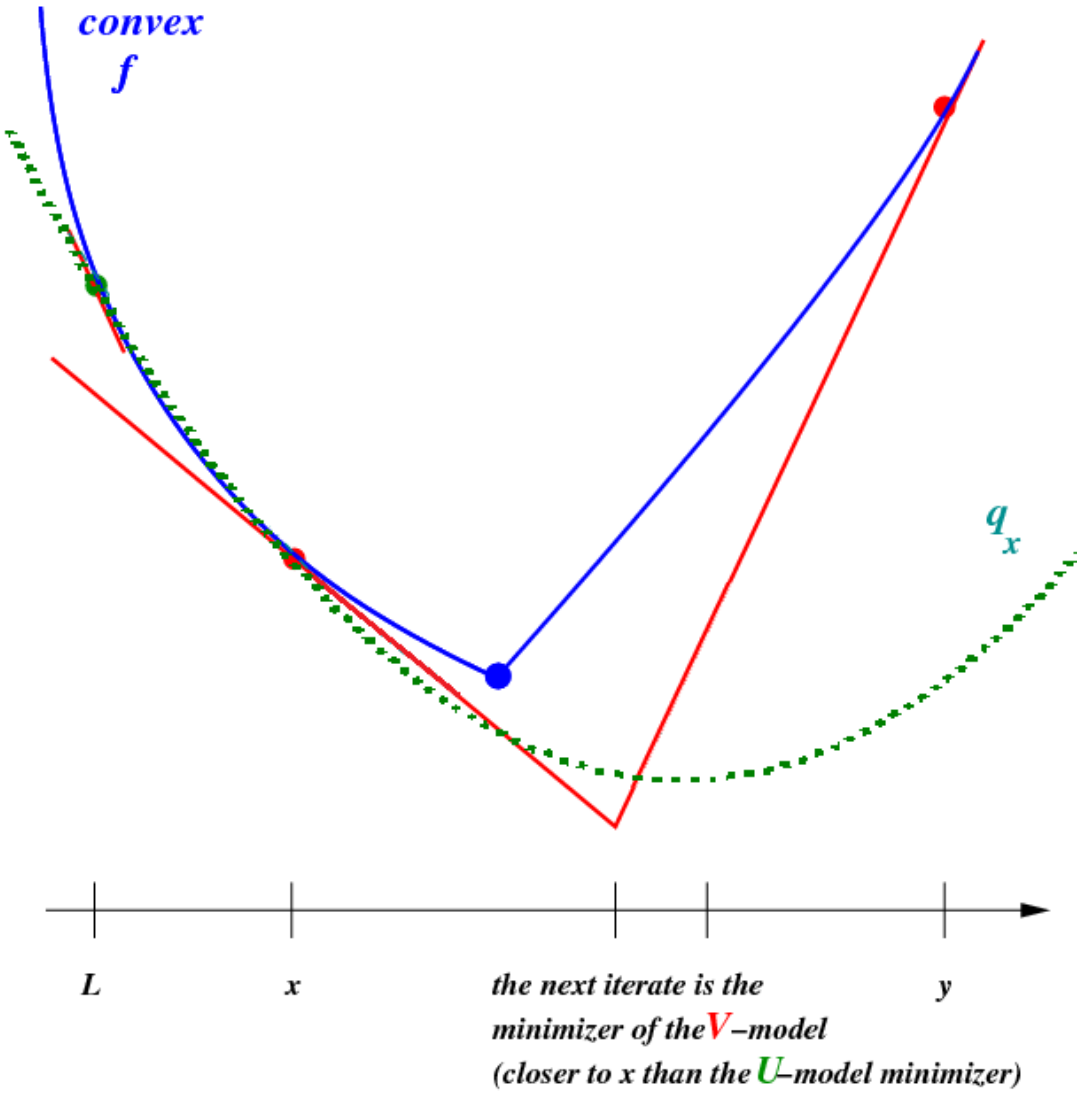
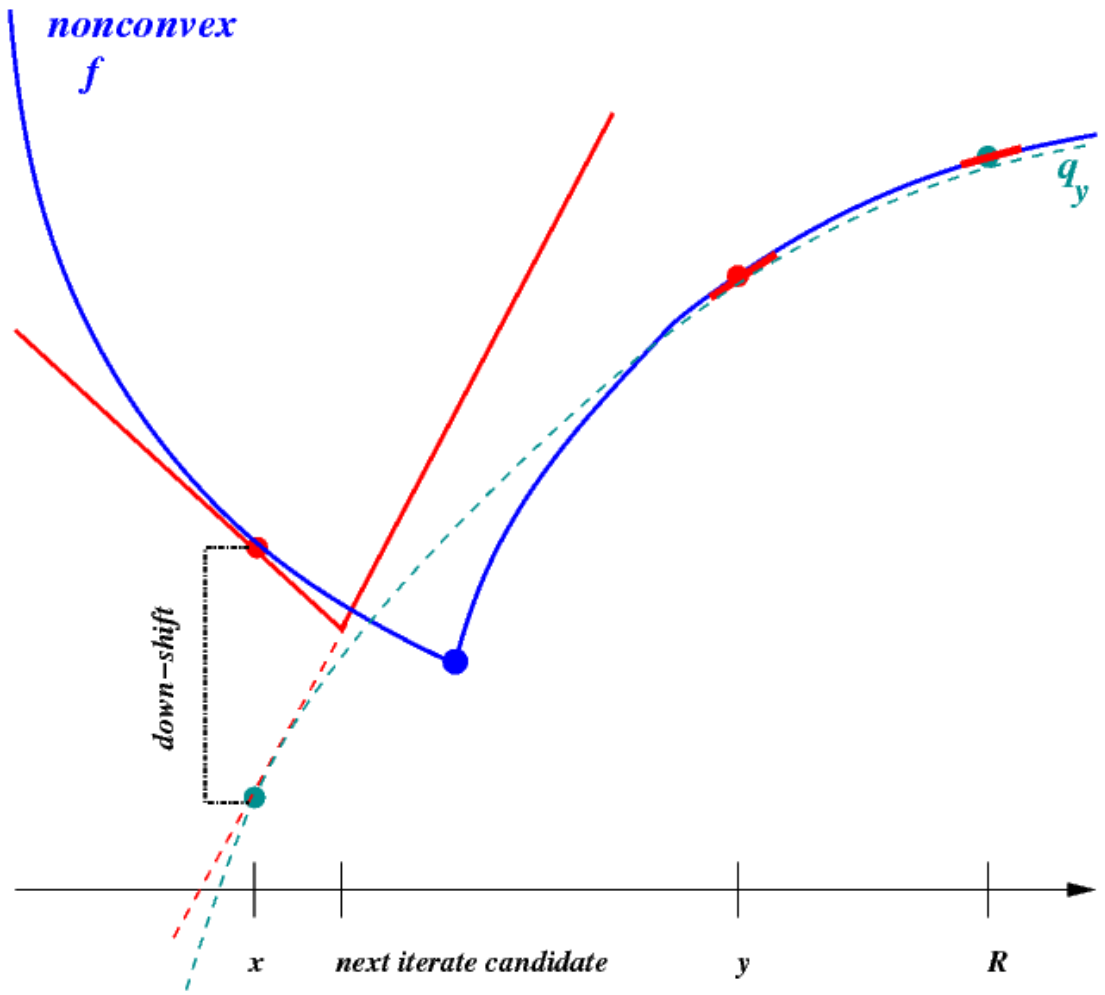
Solving the proximal subproblem with polyhedral M gives $G(x)$ [E(x)] as a convex combination of $\tilde{g}(x, y_i)$ [$\tilde{e}(x, y_i)$].

Geometry of single variable VU-minimization

($n = 1$)





superlinearly convergent for certain piecewise C^2 functions when safeguarded properly

An interval $[x, y]$ or $[y, x]$ is called compatible if $f(x) \leq f(y)$ and $\langle g(x), (y - x) \rangle \leq 0$.
The $n = 1$ algorithm generates such intervals.

However, the viable line search of the n-variable algorithm determines the endpoints of its t-interval of uncertainty based on satisfaction (or not) of an Armijo f-descent test dictated by the serious point definition given below.

For a compatible t-interval the line search defines its next iterate as in the above algorithm; otherwise the next iterate is the bisector of the t-interval.

The $n = 1$ algorithm updates two 2^{nd} derivative estimates and associated $quadratic$ f-approximates using previous interval endpoints.

The n-variable algorithm proceeds similarly, with respect to V-models, using matrices $H(x + td(x))$, updated by an SR1 formula during line search on t, to employ if negative curvature is found.

Specific Viable Model Definition

Given a center x and a bundle point y with associated data $g(y)$, $H(y)$ and $s(y)$ compute the curvature

$$h(x,y) := \left\langle x-y, H(y)(x-y) \right\rangle$$

Nonnegative curvature h:

$\tilde{g}(x,y) := g(y)$ $\qquad$ $\tilde{e}(x,y) := \max(e(x,y), s(y)|x-y|^2)$

Negative curvature h:

$\tilde{g}(x,y) := g(y) + H(y)(x-y)$ $\qquad$ $\tilde{e}(x,y) := \max(e(x,y) - \frac{1}{2}h, s(y)|x-y|^2)$

Specific Viable Model Definition

Given a center x and a bundle point y with associated data $g(y)$, $H(y)$ and $s(y)$ compute the curvature

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$s(y) \in [s_{min}, s_{max}]$
with safeguard $s_{min} > 0$ if f is not convex, to obtain **V2**.

These definitions immediately imply **V1**.

Specific viable null/serious point definitions

The null pont definition is the weakest one known such that infinite number of consecutive null steps with μ nondecreasing and $x_k = \bar{x}$ fixed make the $D_\ell(\bar{x})$-sequence converge to zero (**V3 null version**).

Four parameters for linear combinations of
$D(x)$ and $\mu|d(x)|^2 = |G(x)|^2/\mu = -\langle G(x), d(x)\rangle$,
with bounds for obtaining **V3, V4**.

- null point D-decrease: $m_N \in (0, 1)$
- serious point Armijo-type f-decrease: $m_A \in (0, m_N)$
- serious point small stepsize D-decrease: $m_S \in (0, m_N\text{-}m_A)$
- both points V-model improvement: $m_V \in [0, 1]$
- a fifth parameter is possible for even more serious flexibility

Specific viable null/serious point definitions

$y_+ = x + td(x)$ with $t > 0$

is a null step point if

$$-\tilde{e}(x, y_+) + \langle \tilde{g}(x, y_+), d(x) \rangle \geq -m_N D(x) - m_V \tfrac{1}{2}\mu |d(x)|^2;$$

is a serious step point if

$$[f(y_+) - f(x)]/t \leq -m_A D(x) - m_V \tfrac{1}{2}\mu |d(x)|^2$$

and $t \geq 1$ or $\tilde{e}(x, y_+) \geq m_S D(x)$.

Lemma. If f is convex and $s_{max} = 0$ then $\tilde{g} = g$, $\tilde{e} = e$ and $t = 1$ gives either a null or a serious point.

For $t < 1$ the inequality with parameter m_S ensures **V3** for a serious point sequence when its corresponding t-sequence converges to zero.

Convergence of Specific Viable Algorithm

Theorem(Stationarity).

Suppose f is semismooth and

- the bundle algorithm does not terminate,
- the prox-parameters are in a positive interval $[\mu_{min}, \mu_{max}]$, and
- the sequences of centers $\{x_k\}$ and matrices $\{H(y_\ell)\}$ are bounded

Then any $\bar{x}$ that is a limit point of $\{x_k\}$ is stationary for f.

Proof shows satisfaction of **V1** to **V4** with the asymptotic conditions based on boundedness of $\{y_\ell\}$, which follows from $\{x_k\}$ bounded and **V4** depending on semismoothness of f. $\square$

Future research

(i) For the exceptional case when $y(1) = x + d(x)$ does not satisfy an Armijo descent test, determine conditions for when $H(y(1))$ can be an SR1 update of $H(y_j)$ for some y_j active in the bundle that generated $d(x)$. This would include the angle between $d(x)$ and $y_j - y(1)$ being small.

(ii) Determine choices for $s(y)$ in the n-variable case; dependence on x too as in the 1-variable case?

(iii) Using the above bundle algorithm develop a VU-algorithm for lower-C^2 functions [Janin, 1974], [Rockafellar, 1982].
This involves choosing values for $m_V \leq 1$ to generate very good V-models.

Recall, $y_+ = x + td(x)$ with $t > 0$

is a null step point if

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$$[f(y_+) - f(x)]/t \leq -m_A D(x) - m_V \tfrac{1}{2}\mu |d(x)|^2$$

and $$t \geq 1 \quad \text{or} \quad \tilde{e}(x, y_+) \geq m_S D(x).$$

Line Search with variable t

The search generates a sequence of nested intervals $[t_L, t_R]$ where $x + td(x)$ with $t = t_L(t_R)$ does (does not) satisfy the serious point Armijo f-descent condition.

Start with $t = 1$ in the initial interval $[t_L, t_R) = [0, \infty)$.

If $t = 1 =: t_R$ then enter the interpolation

Loop: If $x + td(x)$ is a serious or null point, exit.

If $[t_L, t_R]$ is a VU-model compatible interval compute the next value of t as in the single variable algorithm.

Else replace t by the bisector of $[t_L, t_R]$.

Replace the A-appropriate endpoint of $[t_L, t_R]$ by t and go to Loop.

Else ($t = 1 =: t_L$),

sequentially increase t until there is an exit with
$t =: t_L$ and $\langle g(x + td(x)), d(x)\rangle$ satisfying a Wolfe test, or
$t =: t_L$ and t being too large (i.e. $f(x + td(x))$ too small),
or
$t =: t_R$.

In the last case an interpolation phase as above could be entered to find a serious point, possibly better than the one given by the interpolation entering t_L value.

$Lemma$. If f is semismooth then the above line search is finite.