2.2 Separable Equations

22. Solve the initial value problem

\[ y' = \frac{3x^3}{3y^3 - 4}, \quad y(1) = 0 \]

and determine the interval in which the solution is valid.

*Hint:* To find the interval of definition, look for points where the integral curve has a vertical tangent.

23. Solve the initial value problem

\[ y' = 2y^2 + xy, \quad y(0) = 1 \]

and determine where the solution attains its minimum value.

24. Solve the initial value problem

\[ y' = \frac{(2-y)}{(3+2y)}, \quad y(0) = 0 \]

and determine where the solution attains its maximum value.

25. Solve the initial value problem

\[ y' = \frac{2 \cos 2x}{3+2y}, \quad y(0) = -1 \]

and determine where the solution attains its maximum value.

26. Solve the initial value problem

\[ y' = 2(1+x)(1+y^3), \quad y(0) = 0 \]

and determine where the solution attains its minimum value.

27. Consider the initial value problem

\[ y' = ty(4-y)/3, \quad y(0) = y_0. \]

(a) Determine how the behavior of the solution as \( t \) increases depends on the initial value \( y_0 \).

(b) Suppose that \( y_0 = 0.5 \). Find the time \( T \) at which the solution first reaches the value 3.98.

28. Consider the initial value problem

\[ y' = ty(4-y)/(1+t), \quad y(0) = y_0 > 0. \]

(a) Determine how the solution behaves as \( t \rightarrow \infty \).

(b) If \( y_0 = 2 \), find the time \( T \) at which the solution first reaches the value 3.99.

(c) Find the range of initial values for which the solution lies in the interval \( 3.99 < y < 4.01 \) by the time \( t = 2 \).

29. Solve the equation

\[ \frac{dy}{dx} = ay + b \quad \frac{cy + d}{cx + d}, \]

where \( a, b, c, \) and \( d \) are constants.

**Homogeneous Equations.** If the right side of the equation \( \frac{dy}{dx} = f(x, y) \) can be expressed as a function of the ratio \( y/x \) only, then the equation is said to be
PROBLEMS
In each of Problems 1 through 12:
(a) Draw a direction field for the given differential equation.
(b) Based on an inspection of the direction field, describe how solutions behave for large t.
(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as \( t \to \infty \).

1. \( y' + 3y = t + e^{-3t} \)
2. \( y' - 2y = t^2 e^{2t} \)
3. \( y' + y = te^{-t} + 1 \)
4. \( y' + (1/t)y = 3 \cos 2t, \quad t > 0 \)
5. \( y' - 2y = 3e^t \)
6. \( ty' + 2y = \sin t, \quad t > 0 \)
7. \( y' + 2ty = 2te^{-t} \)
8. \( (1 + t^2)y' + 4ty = (1 + t^2)^{-2} \)
9. \( 2y' + y = 3t \)
10. \( ty' - y = t^2 e^{-t}, \quad t > 0 \)
11. \( y' + y = 5 \sin 2t \)
12. \( 2y' + y = 3t^2 \)

In each of Problems 13 through 20, find the solution of the given initial value problem.
13. \( y' - y = 2te^{2t}, \quad y(0) = 1 \)
14. \( y' + 2y = te^{-2t}, \quad y(1) = 0 \)
15. \( ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0 \)
16. \( y' + (2/t)y = (\cos t)/t^2, \quad y(x) = 0, \quad t > 0 \)
17. \( y' - 2y = e^{2t}, \quad y(0) = 3 \)
18. \( ty' + 2y = \sin t, \quad y(x/2) = 1, \quad t > 0 \)
19. \( t^2 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0, \quad t < 0 \)
20. \( ty' + (t + 1)y = t, \quad y(\ln 2) = 1, \quad t > 0 \)

In each of Problems 21 through 23:
(a) Draw a direction field for the given differential equation. How do solutions appear to behave as \( t \) becomes large? Does the behavior depend on the choice of the initial value \( a \)? Let \( a_0 \) be the value of \( a \) for which the transition from one type of behavior to another occurs. Estimate the value of \( a_0 \).
(b) Solve the initial value problem and find the critical value \( a_0 \) exactly.
(c) Describe the behavior of the solution corresponding to the initial value \( a_0 \).

21. \( y' - \frac{1}{2}y = 2 \cos t, \quad y(0) = a \)
22. \( 2y' + y = e^{t^3}, \quad y(0) = a \)
23. \( 3y' - 2y = e^{-\sqrt{t}}, \quad y(0) = a \)

In each of Problems 24 through 26:
(a) Draw a direction field for the given differential equation. How do solutions appear to behave as \( t \to 0^+ \)? Does the behavior depend on the choice of the initial value \( a \)? Let \( a_0 \) be the value of \( a \) for which the transition from one type of behavior to another occurs. Estimate the value of \( a_0 \).
(b) Solve the initial value problem and find the critical value \( a_0 \) exactly.
(c) Describe the behavior of the solution corresponding to the initial value \( a_0 \).

24. \( ty' + (t + 1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0 \)
25. \( ty' + 2y = (\sin t)/t, \quad y(\pi/2) = a, \quad t < 0 \)
26. \( (\sin t)y' + (\cos t)y = e^t, \quad y(1) = a, \quad 0 < t < \pi \)
27. Consider the initial value problem
\[ y' + \frac{1}{2}y = 2 \cos t, \quad y(0) = -1. \]
Find the coordinates of the first local maximum point of the solution for \( t > 0 \).
28. Consider the initial value problem
\[ y' + \frac{1}{2}y = 1 - \frac{1}{2}t, \quad y(0) = y_0. \]
Find the value of \( y_0 \) for which the solution touches, but does not cross, the \( t \)-axis.
1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

2. A tank initially contains 120 L of pure water. A mixture containing a concentration of \( \gamma \) g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of \( \gamma \) for the amount of salt in the tank at any time \( t \). Also find the limiting amount of salt in the tank as \( t \to \infty \).

3. A tank originally contains 100 gal of fresh water. Then water containing \( \frac{1}{3} \) lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

4. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

5. A tank contains 100 gal of water and 50 oz of salt. Water containing a salt concentration of \( \frac{1}{4}(1 + \frac{1}{2} \sin t) \) oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.
   (a) Find the amount of salt in the tank at any time.
   (b) Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.
   (c) The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

20. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.
   (a) Find the maximum height above the ground that the ball reaches.
   (b) Assuming that the ball misses the building on the way down, find the time that it hits the ground.
   (c) Plot the graphs of velocity and position versus time.

21. Assume that the conditions are as in Problem 20 except that there is a force due to air resistance of magnitude \( |v|/30 \) directed opposite to the velocity, where the velocity \( v \) is measured in m/s.
   (a) Find the maximum height above the ground that the ball reaches.
   (b) Find the time that the ball hits the ground.
   (c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 20.
PROBLEMS

In each of Problems 1 through 6, determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. \((t - 3)y' + (\ln t)y = 2t, \quad y(1) = 2\)
2. \(t(t - 4)y' + y = 0, \quad y(2) = 1\)
3. \(y' + (\tan t)y = \sin t, \quad y(\pi) = 0\)
4. \((4 - t^2)y' + 2ty = 3t^2, \quad y(-3) = 1\)
5. \((4 - t^2)y' + 2ty = 3t^2, \quad y(1) = -3\)
6. \((\ln t)y' + y = \cot t, \quad y(2) = 3\)

In each of Problems 7 through 12, state where in the \(ty\)-plane the hypotheses of Theorem 2.4.2 are satisfied.

7. \(y' = \frac{t - y}{2t + 5y}\)
8. \(y' = (1 - t^2 - y^2)^{1/2}\)
9. \(y' = \frac{\ln |ty|}{1 - t^2 + y^2}\)
10. \(y' = (t^2 + y^2)^{1/2}\)
11. \(\frac{dy}{dt} = \frac{1 + t^2}{3y - y^2}\)
12. \(\frac{dy}{dt} = \frac{(\cot t)y}{1 + y}\)

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value \(y_0\).

13. \(y' = -4t/y, \quad y(0) = y_0\)
14. \(y' = 2ty^2, \quad y(0) = y_0\)
15. \(y' + y^3 = 0, \quad y(0) = y_0\)
16. \(y' = t^2/y(1 + t^2), \quad y(0) = y_0\)

In each of Problems 17 through 20, draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as \(t\) increases and how their behavior depends on the initial value \(y_0\) when \(t = 0\).

17. \(y' = ty(3 - y)\)
18. \(y' = y(3 - ty)\)
19. \(y' = -y(3 - ty)\)
20. \(y' = t - 1 - y^2\)

21. Consider the initial value problem \(y' = y^{1/2}, y(0) = 0\) from Example 3 in the text.
   (a) Is there a solution that passes through the point \((1, 1)\)? If so, find it.
   (b) Is there a solution that passes through the point \((2, 1)\)? If so, find it.
   (c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at \(t = 2\).