Homogeneous Equations. If the right side of the equation \( dy/dx = f(x, y) \) can be expressed as a function of the ratio \( y/x \) only, then the equation is said to be homogeneous. Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

30. Consider the equation

\[
\frac{dy}{dx} = \frac{y - 4x}{x - y}. \tag{i}
\]

(a) Show that Eq. (i) can be rewritten as

\[
\frac{dy}{dx} = \frac{y/x - 4}{1 - (y/x)}, \tag{ii}
\]

thus Eq. (i) is homogeneous.

(b) Introduce a new dependent variable \( v \) so that \( v = y/x \), or \( y = xv(x) \). Express \( dy/dx \) in terms of \( x, v, \) and \( dv/dx \).

(c) Replace \( y \) and \( dy/dx \) in Eq. (ii) by the expressions from part (b) that involve \( v \) and \( dv/dx \). Show that the resulting differential equation is

\[
v + \frac{dv}{dx} = \frac{v - 4}{1 - v},
\]

or

\[
\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \tag{iii}
\]

Observe that Eq. (iii) is separable.

(d) Solve Eq. (iii), obtaining \( v \) implicitly in terms of \( x \).

(e) Find the solution of Eq. (i) by replacing \( v \) by \( y/x \) in the solution in part (d).

(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (i) actually depends only on the ratio \( y/x \). This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution \( y = xv(x) \) transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing \( v \) by \( y/x \) gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.

(b) Solve the differential equation.

(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

31. \[
\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \tag{31}
\]
32. \[
\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} \tag{32}
\]
33. \[
\frac{dy}{dx} = \frac{4y - 3x}{2x - y} \tag{33}
\]
34. \[
\frac{dy}{dx} = \frac{4x + 3y}{2x + y} \tag{34}
\]
35. \[
\frac{dy}{dx} = \frac{x + 3y}{x - y} \tag{35}
\]
36. \[
(x^2 + 3xy + y^2) \, dx - x^3 \, dy = 0 \tag{36}
\]
37. \[
\frac{dy}{dx} = \frac{x^2 - 3y^2}{2xy} \tag{37}
\]
38. \[
\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \tag{38}
\]